Phase transitions in the Kuramoto model

Lasko Basnarkov*
Faculty of Electrical Engineering and Information Technologies, SS. Cyril and Methodius University, P.O. Box 574, Skopje, Macedonia

Viktor Urumov
Faculty of Natural Sciences and Mathematics, SS. Cyril and Methodius University, P.O. Box 162, Skopje, Macedonia
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We consider the Kuramoto model of phase oscillators with natural frequencies distributed according to a unimodal function with the plateau section in the middle representing the maximum and symmetric tails falling off predominantly as $|\omega-\omega_0|^m$, $m>0$, in the vicinity of the flat region. It is found that the phase transition is of first order as long as there is a finite flat region and that in the vicinity of the critical coupling the following scaling law holds $r-r_c \propto (K-K_c)^{2/(2m+3)}$, where r is the order parameter and K is the coupling strength of the interacting oscillators.

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Synchronization is a ubiquitous cooperative phenomenon of interacting units. Unison chirping of crickets, synchronous flashing of fireflies in some forests in South Asia, and networks of pacemaker cells in the heart are natural examples, while arrays of lasers and Josephson junctions are physical systems that show synchronization phenomena [1]. The most successful and mathematically most tractable model for describing synchronization is the Kuramoto model [2]. An excellent description of the model and its extensions, as well as some of its applications can be found in a recent review paper [3]. The model describes synchronization of phase oscillators with different natural frequencies with all-to-all interaction. It was found that when coupling strength exceeds some critical value, part of the oscillators lock to a common frequency. Such a state can be suitably described with an order parameter measuring the degree of coherence of the phases of the oscillators. For general symmetric distributions with single maximum and with nonzero second derivative at the maximum, the phase transition is of second-order type [2]. The phase transition is of first-order type when the distribution of natural frequencies is uniform [4]. We extend that result to all distributions having a plateau at the maximum. Also we obtain different scaling behavior near the transition point for different classes of functions.

The Kuramoto model considers the population of N phase oscillators with phases θ_i evolving by the equation

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \tag{1}$$

with equal coupling strength K/N between all pairs of oscillators i and j. The natural frequencies of the oscillators ω_i are drawn from some distribution $g(\omega)$. We will consider distributions that are symmetric around the mean [which can be taken zero without loss of generality, $g(-\omega)=g(\omega)$], and also nonincreasing for $\omega>0$. The degree of coherence of phases is suitably expressed by the order parameter r defined as

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}. \tag{2}$$

Because of symmetry, for an infinite number of oscillators the mean phase can be also taken to zero. Then the dynamics of every oscillator is governed by its natural frequency and its coupling to the mean phase

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi-\theta_i) = \omega_i - Kr\sin\theta_i. \tag{3}$$

For an infinite population of oscillators one can analyze the probability density function $\rho(\theta,\omega,t)$ of oscillators having intrinsic frequency ω and phase θ. Its evolution is governed by the continuity equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial[\rho v]}{\partial\theta} = \frac{\partial\rho}{\partial t} + \frac{\partial[\rho(\omega - Kr\sin\theta)]}{\partial\theta} = 0, \tag{4}$$

where we have omitted the subscript i appearing in Eq. (3) for the speed $v=\dot{\theta}$. Then the order parameter is expressed through the probability density function as

$$r = \int\int d\theta d\omega e^{i\theta} g(\omega)\rho(\theta,\omega,t). \tag{5}$$

We are interested in stationary solutions. For strong enough coupling a nonzero solution for the order parameter r is possible. Then oscillators with natural frequencies obeying $|\omega|<Kr$ will be frequency locked to the mean phase, and their phase will be

$$\theta = \arcsin\left(\frac{\omega}{Kr}\right). \tag{6}$$

The locked phase in Eq. (6) is constrained to $-\pi/2 \le \theta \le \pi/2$ because the other solutions are unstable. The stationary distribution describing locked oscillators is

$$\rho(\theta,\omega) = \delta\left(\theta - \arcsin\left(\frac{\omega}{Kr}\right)\right). \tag{7}$$

The other oscillators run out of synchrony and their distribution is symmetric and they do not contribute to the order

*lasko@feit.ukim.edu.mk

parameter. The self-consistent equation for the order parameter r is

$$r=\int_{|\omega|<Kr}\int_{-\pi/2}^{\pi/2}d\theta d\omega e^{i\theta}g(\omega)\delta\left(\theta-\arcsin\left(\frac{\omega}{Kr}\right)\right). \quad (8)$$

Because r is real, the imaginary part of the integral is zero, so

$$r=\int_{|\omega|<Kr}\int_{-\pi/2}^{\pi/2}d\theta d\omega \cos\theta g(\omega)\delta\left(\theta-\arcsin\left(\frac{\omega}{Kr}\right)\right). \quad (9)$$

Integrating the last equation and using the properties of the Dirac function and the symmetry of $g(\omega)$ one obtains

$$r=2Kr\int_0^{\pi/2}d\theta\cos^2\theta g(Kr\sin\theta). \quad (10)$$

The trivial solution $r=0$ corresponds to the incoherence and is the only one for coupling smaller than critical. The nonzero solution is related to the partial synchronization. If the distribution is unimodal (nonincreasing for $\omega>0$, and with maximum only for $\omega=0$) the critical coupling is obtained by taking the limit $r\to 0$ in Eq. (10),

$$1=2K\int_0^{\pi/2}d\theta\cos^2\theta g(0), \quad (11)$$

and its value is

$$K_c=\frac{2}{\pi g(0)}. \quad (12)$$

The phase transition to partial synchronization is of second-order type and power series expansion of the unimodal distribution $g(w)$ around the zero leads to the following scaling behavior [3]:

$$r\sim\sqrt{\frac{-16(K-K_c)}{\pi K_c^4 g''(0)}}. \quad (13)$$

A more general result has been obtained for the case

$$g(\omega)\sim 1-|\omega-\omega_0|^m,\quad m>0, \quad (14)$$

leading to the following dependence for the order parameter

$$r\sim|K-K_c|^{1/m} \quad (15)$$

in the vicinity of the critical point [5].

Here we consider a symmetric distribution $g(\omega)=g(-\omega)$ with a flat maximum. For $\omega>0$, $g(\omega)$ is defined by

$$g(\omega)=g(0)+f(\omega-\omega_0)H(\omega-\omega_0), \quad (16)$$

where $\omega_0>0$, $H(\omega)$ is the unit step Heaviside function, and $f(\omega)$ is a decreasing function of ω with $f(0)=0$. The tail of $g(\omega)$ is an arbitrary function with finite or infinite support with the restrictions imposed by the fact that $g(\omega)$ is nonnegative, nonincreasing, and normalizable. As we will see later, the critical point is determined by the width of the plateau. The scaling behavior of the order parameter near the critical point (K_c,r_c) is determined only from the oscillators that are locked to the mean phase (those from the plateau and its vicinity). Therefore, in order to examine the asymptotic behavior close to the critical point, we can keep only the dominant term of the tail $f(\omega-\omega_0)$, for which we assume a power law dependence. For the purpose of determining the asymptotics, we take that above and close to the flat region, $g(\omega)$ behaves essentially as

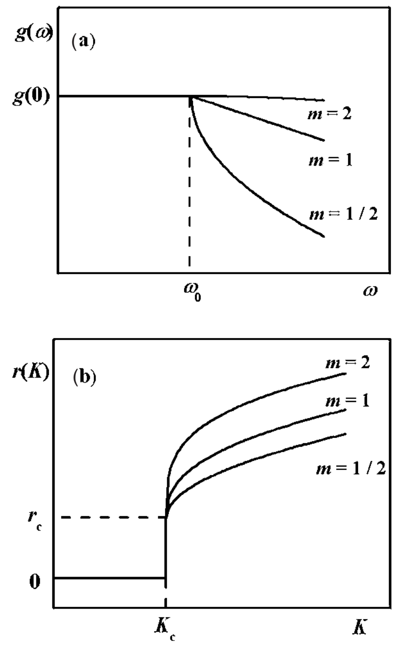

FIG. 1. (a) Sketch of distributions [Eq. (17)] for positive frequencies ω near the end point of the plateau for $m=1/2$, 1, and 2. (b) The order parameter $r(K)$ for the distributions given in (a).

$$g(\omega)=g(0)-C(\omega-\omega_0)^m H(\omega-\omega_0), \quad (17)$$

where C is a constant and $m>0$ is a parameter. Schematically, $g(\omega)$ is depicted in Fig. 1(a).

From Eq. (10) we see that the border of frequencies of oscillators contributing to the order parameter is $\omega_b=Kr$. If $\omega_b\leq\omega_0$ the equation for r is the same as the one for obtaining the critical coupling [Eq. (11)], which is consistent only for $K=K_c$ (the only solution for $K<K_c$ is $r=0$). So, for the same value of the coupling—the critical one, the interval of the frequencies belonging to synchronized oscillators can stretch as far as all the plateau. For different frequencies ω_b one has different values for the order parameter $r=\omega_b/K_c$, which can be deduced from Eq. (6) by taking $\theta=\pi/2$. All such values for r are admissible solutions of the self-consistent equation (10) and correspond to the vertical stretch of solutions depicted in Fig. 1(b). If all the oscillators belonging to the plateau are locked we obtain the critical value of the order parameter

$$r_c=\frac{\omega_0}{K_c}=\frac{\pi\omega_0 g(0)}{2}, \quad (18)$$

which for uniform distribution of natural frequencies has a value $r_c=\pi/4$ [4]. The value of r_c decreases with narrowing of the plateau, and in the limiting case $\omega_0 \to 0$, the phase transition becomes of second order.

For coupling stronger than the critical, some oscillators in the vicinity of the plateau lock too. Then the self-consistent equation for r after elimination of one r on each side in Eq. (10) and using the form of the distribution $g(\omega)$ [Eq. (17)], becomes

$$1 = 2K\int_0^{\pi/2} d\theta \cos^2 \theta g(0) - 2CK\int_0^{\pi/2} d\theta \cos^2 \theta (Kr)^m \times (\sin \theta - \sin \theta_0)^m H[Kr(\sin \theta - \sin \theta_0)], \tag{19}$$

where phases θ and θ_0 correspond to frequencies ω and ω_0 according to Eq. (6). Using Eq. (12) the first integral is simply

$$I_1 = 2K\int_0^{\pi/2} d\theta \cos^2 \theta g(0) = \frac{K}{K_c}. \tag{20}$$

Because of the Heaviside function in Eq. (17), the second integral runs from $\theta_0 = \pi/2 - \delta\theta$ to $\pi/2$,

$$I_2 = 2CK(Kr)^m \int_{\theta_0}^{\pi/2} d\theta \cos^2 \theta (\sin \theta - \sin \theta_0)^m. \tag{21}$$

When the coupling is little bigger than the critical, then the highest locked frequency is close to ω_0, and so the corresponding phase θ_0 is close to $\pi/2$, or $\delta\theta \to 0$. After some trigonometric transformations, using the dominant terms of sine and cosine functions from their power series and an obvious variable substitution, I_2 becomes

$$I_2 = 2CK(Kr)^m \delta\theta^{2m+3} \int_0^1 dx(1-x)^2 x^m \left(1 - \frac{x}{2}\right)^m = A\,\delta\theta^{2m+3}. \tag{22}$$

The integral in I_2 is convergent for $m > -1$. For integer values of the parameter m, the integral is reduced to

$$I_2 = 2CK(Kr)^m (\delta\theta)^{2m+3} S_m, \tag{23}$$

where S_m is the sum

$$S_m = \sum_{k=0}^{m} \frac{(-1)^{m-k} 2^{k-m+1} C_k^m}{(2m-k+1)(2m-k+2)(2m-k+3)}, \tag{24}$$

involving binomial coefficients C_k^m. The first few values of the sum are $S_0 = 1/3$, $S_1 = 1/15$, and $S_2 = 2/105$.

Using Eqs. (20) and (22) and (22) in the self-consistency equation (19) one obtains

$$1 = \frac{K}{K_c} - A(\delta\theta)^{2m+3}. \tag{25}$$

Near critical point (K_c, r_c),

$$r = r_c + \delta r,$$

$$K = K_c + \delta K, \tag{26}$$

one can take

$$A \approx 2CK_c(K_c r_c)^m \int_0^1 dx(1-x)^2 x^m \left(1 - \frac{x}{2}\right)^m, \tag{27}$$

which means we can treat A as a constant. Then the differentials $\delta\theta$ and δK are related with

$$\delta K = AK_c(\delta\theta)^{2m+3}. \tag{28}$$

For couplings a bit stronger than the critical K_c, the boundary of the plateau and the corresponding phase are related with [Eq. (6)],

$$\omega_0 = (K_c + \delta K)(r_c + \delta r)\sin \theta_0. \tag{29}$$

Using the substitution $\delta\theta = \pi/2 - \theta_0$ and Eq. (18) one can obtain another equation relating the differentials $\delta\theta$, δK, and δr,

$$\delta\theta \approx \sqrt{\frac{4}{\pi\omega_0 g(0)}\delta r + \pi g(0)\delta K}, \tag{30}$$

which for uniform distribution reduces to [4]

$$\delta\theta \approx \sqrt{\frac{8}{\pi}\delta r + \frac{\pi}{2\omega_0}\delta K}. \tag{31}$$

Finally, combining Eqs. (28) and (30) leads to

$$\delta K^{2/(2m+3)} \sim C_1 \delta K + C_2 \delta r, \tag{32}$$

from where it is clear that δr scales as

$$\delta r \sim \delta K^{2/(2m+3)}. \tag{33}$$

The dependence $r(K)$ for several values of m is shown in Fig. 1(b). As can be seen this general result also contains the previously reported scaling for uniform distribution ($m=0$) of natural frequencies [4]. Usually one defines critical exponents describing the behavior of physical parameters (quantities) in the immediate neighborhood of critical points or second-order phase transitions. Here we have a peculiar situation with singular behavior near first-order phase transition. In addition one can remark that the exponents obtained in this case differ from those characterizing the second-order transition in the Kuramoto model [Eq. (15)].

Synchronization of two interacting phase oscillators takes place if the coupling is strong enough, or their natural frequencies are sufficiently close [6]. For a population of equally coupled oscillators, with an increase of the coupling strength synchronization appears first among the closest oscillators. When the distribution function has a single maximum the likelihood of synchronization is largest among the oscillators with natural frequencies leading to highest values of $g(\omega)$. With the coupling constant increasing toward the critical value, the seed of the cluster of synchronized oscillators consists of an infinitesimal interval around the maximum, and the phase transition is of second order. However, for distribution functions with a flat top there is no such "dominant" density and all the plateau forms the seed for the synchronized cluster. Then a macroscopic part of the oscillators contribute to the order parameter, thus leading to a first-order transition.

In this Brief Report we have studied the Kuramoto model with natural frequencies distributed according to the sketches in Fig. 1(a). From previous studies [7] it is known that in the Kuramoto model with bimodal distribution of natural frequencies, the phase transition is of first order showing typical hysteresis and metastability. The case considered here is in some sense intermediate between bimodal and unimodal distributions. The distribution function in the interval between the two maxima of the bimodal distribution is represented by a flat region instead of a distribution with single minimum. The disappearance of the minimum leads to destruction of the competition between surrounding maxima, squeezing the hysteresis into a single vertical line in the $r(K)$ dependence. As a result the model under consideration lacks hysteresis and metastability.

An explicit asymptotic dependence was obtained for the order parameter above the critical value of the coupling constant. It is intuitively quite acceptable that the increase of the order parameter is faster when the power law decay of the natural frequencies is less pronounced.

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